

Project 1

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(Dated: September 12, 2023)

<https://github.uio.no/FYS3150-G2-2023/Project-1>

PROBLEM 1

First, we rearrange the equation.

$$\begin{aligned}-\frac{d^2u}{dx^2} &= 100e^{-10x} \\ \frac{d^2u}{dx^2} &= -100e^{-10x}\end{aligned}$$

Now we find $u(x)$.

$$\begin{aligned}u(x) &= \int \int \frac{d^2u}{dx^2} dx^2 \\ &= \int \int -100e^{-10x} dx^2 \\ &= \int \frac{-100e^{-10x}}{-10} + c_1 dx \\ &= \int 10e^{-10x} + c_1 dx \\ &= \frac{10e^{-10x}}{-10} + c_1x + c_2 \\ &= -e^{-10x} + c_1x + c_2\end{aligned}$$

Using the boundary conditions, we can find c_1 and c_2

$$\begin{aligned}u(0) &= 0 \\ -e^{-10 \cdot 0} + c_1 \cdot 0 + c_2 &= 0 \\ -1 + c_2 &= 0 \\ c_2 &= 1\end{aligned}$$

$$\begin{aligned}u(1) &= 0 \\ -e^{-10 \cdot 1} + c_1 \cdot 1 + c_2 &= 0 \\ -e^{-10} + c_1 + c_2 &= 0 \\ c_1 &= e^{-10} - c_2 \\ c_1 &= e^{-10} - 1\end{aligned}$$

Using the values that we found for c_1 and c_2 , we get

$$\begin{aligned}u(x) &= -e^{-10x} + (e^{-10} - 1)x + 1 \\ &= 1 - (1 - e^{-10})x - e^{-10x}\end{aligned}$$

PROBLEM 2

The code for generating the points and plotting them can be found under.

Point generator code: <https://github.uio.no/FYS3150-G2-2023/Project-1/blob/main/src/analyticPlot.cpp>

Plotting code: <https://github.uio.no/FYS3150-G2-2023/Project-1/blob/main/src/analyticPlot.py>

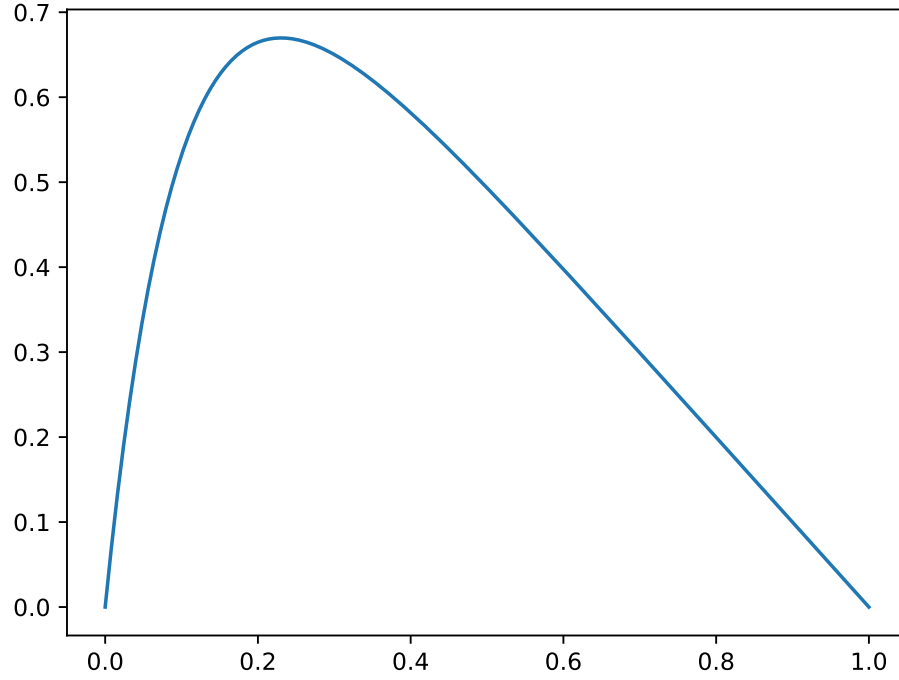


FIG. 1. Plot of the analytical solution $u(x)$.

PROBLEM 3

To derive the discretized version of the Poisson equation, we first need the Taylor expansion for $u(x)$ around x for $x + h$ and $x - h$.

$$u(x + h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + \mathcal{O}(h^4)$$

$$u(x - h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + \mathcal{O}(h^4)$$

If we add the equations above, we get this new equation:

$$\begin{aligned} u(x + h) + u(x - h) &= 2u(x) + u''(x)h^2 + \mathcal{O}(h^4) \\ u(x + h) - 2u(x) + u(x - h) + \mathcal{O}(h^4) &= u''(x)h^2 \\ u''(x) &= \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} + \mathcal{O}(h^2) \\ u''_i(x) &= \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \mathcal{O}(h^2) \end{aligned}$$

We can then replace $\frac{d^2 u}{dx^2}$ with the RHS (right-hand side) of the equation:

$$\begin{aligned} -\frac{d^2 u}{dx^2} &= f(x) \\ \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + \mathcal{O}(h^2) &= f_i \end{aligned}$$

And lastly, we leave out $\mathcal{O}(h^2)$ and change u_i to v_i to differentiate between the exact solution and the approximate solution, and get the discretized version of the equation:

$$\frac{-v_{i+1} + 2v_i - v_{i-1}}{h^2} = 100e^{-10x_i}$$

PROBLEM 4

The value of $u(x_0)$ and $u(x_1)$ is known, using the discretized equation we solve for $\vec{v}$. This will result in a set of equations

$$\begin{aligned} -v_0 + 2v_1 - v_2 &= h^2 \cdot f_1 \\ -v_1 + 2v_2 - v_3 &= h^2 \cdot f_2 \\ &\vdots \\ -v_{m-2} + 2v_{m-1} - v_m &= h^2 \cdot f_{m-1} \end{aligned}$$

where $v_i = v(x_i)$ and $f_i = f(x_i)$. Rearranging the first and last equation, moving terms of known boundary values to the RHS

$$\begin{aligned} 2v_1 - v_2 &= h^2 \cdot f_1 + v_0 \\ -v_1 + 2v_2 - v_3 &= h^2 \cdot f_2 \\ &\vdots \\ -v_{m-2} + 2v_{m-1} &= h^2 \cdot f_{m-1} + v_m \end{aligned}$$

We now have a number of linear equations, corresponding to the number of unknown values, which can be represented as an augmented matrix

$$\left[\begin{array}{cccc|c} 2v_1 & -v_2 & 0 & \dots & 0 & g_1 \\ -v_1 & 2v_2 & -v_3 & 0 & & g_2 \\ 0 & -v_2 & 2v_3 & -v_4 & & g_2 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & & & -v_{m-2} & 2v_{m-1} & g_{m-1} \end{array} \right]$$

Since the boundary values are equal to 0 the RHS can be renamed $g_i = h^2 f_i$ for all i . An augmented matrix can be represented as $\mathbf{A}\vec{x} = \vec{b}$. In this case $\mathbf{A}$ is the coefficient matrix with a tridiagonal signature $(-1, 2, -1)$ and dimension $n \times n$, where $n = m - 2$.

PROBLEM 5

a & b)

$n = m - 2$ since when solving for $\vec{v}$, we are finding the solutions for all the points that are in between the boundaries and not the boundaries themselves. $\vec{v}^*$ on the other hand includes the boundary points.

PROBLEM 6

a)

Renaming the sub-, main-, and supdiagonal of matrix A

$$\begin{aligned}\vec{a} &= [a_2, a_3, \dots, a_{n-1}, a_n] \\ \vec{b} &= [b_1, b_2, b_3, \dots, b_{n-1}, b_n] \\ \vec{c} &= [c_1, c_2, c_3, \dots, c_{n-1}]\end{aligned}$$

Following Thomas algorithm for gaussian elimination, we first perform a forward sweep followed by a backward sweep to obtain $\vec{v}$

Algorithm 1 General algorithm

procedure FORWARD SWEEP($\vec{a}, \vec{b}, \vec{c}$)

$n \leftarrow \text{length of } \vec{b}$

$\vec{b}, \vec{g} \leftarrow \text{vectors of length } n.$

$\hat{b}_1 \leftarrow b_1$

▷ Handle first element in main diagonal outside loop

$\hat{g}_1 \leftarrow g_1$

for $i = 2, 3, \dots, n$ **do**

$d \leftarrow \frac{a_i}{\hat{b}_{i-1}}$

▷ Calculating common expression

$\hat{b}_i \leftarrow b_i - d \cdot c_{i-1}$

$\hat{g}_i \leftarrow g_i - d \cdot \hat{g}_{i-1}$

return $\vec{b}, \vec{g}$

procedure BACKWARD SWEEP($\vec{b}, \vec{g}$)

$n \leftarrow \text{length of } \vec{b}$

$\vec{v} \leftarrow \text{vector of length } n.$

$v_n \leftarrow \frac{g_n}{b_n}$

for $i = n - 1, n - 2, \dots, 1$ **do**

$v_i \leftarrow \frac{g_i - c_i \cdot v_{i+1}}{b_i}$

return $\vec{v}$

b)

Counting the number of FLOPs for the general algorithm by looking at one procedure at a time. For every iteration of i in forward sweep we have 1 division, 2 multiplications, and 2 subtractions, resulting in $5(n - 1)$ FLOPs. For backward sweep we have 1 division, and for every iteration of i we have 1 subtraction, 1 multiplication, and 1 division, resulting in $3(n - 1) + 1$ FLOPs. Total FLOPs for the general algorithm is $8(n - 1) + 1$.

PROBLEM 7

a)

The code can be found at <https://github.uio.no/FYS3150-G2-2023/Project-1/blob/main/src/generalAlgorithm.cpp>

b)

Increasing the number of steps results in an approximation close to the exact solution.

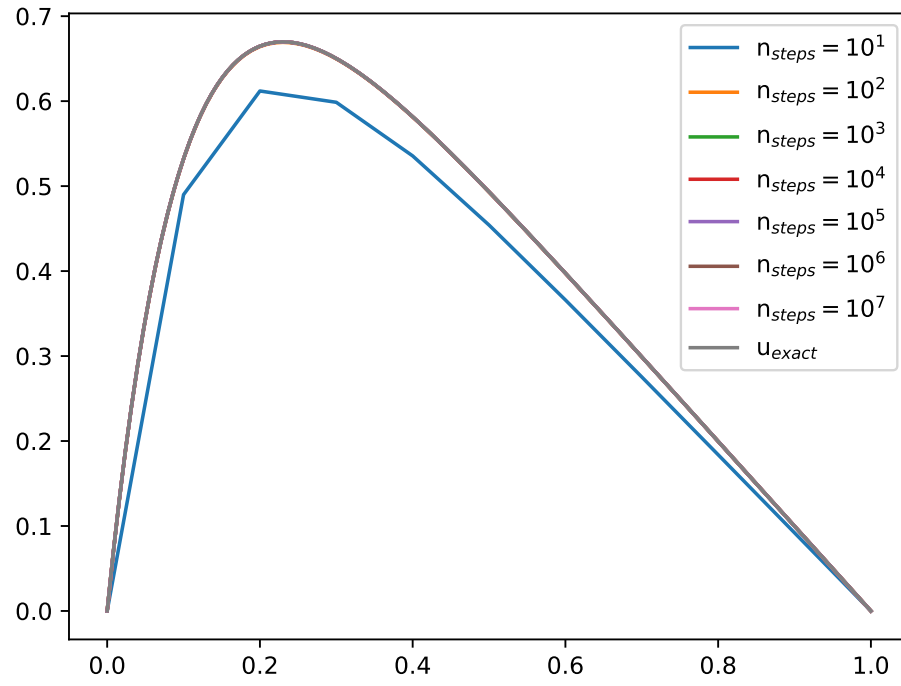


FIG. 2. Plot showing the numeric solution of u_{approx} for n_{steps} and the exact solution u_{exact} .

PROBLEM 8

a)

Increasing number of steps result in a decrease of absolute error.

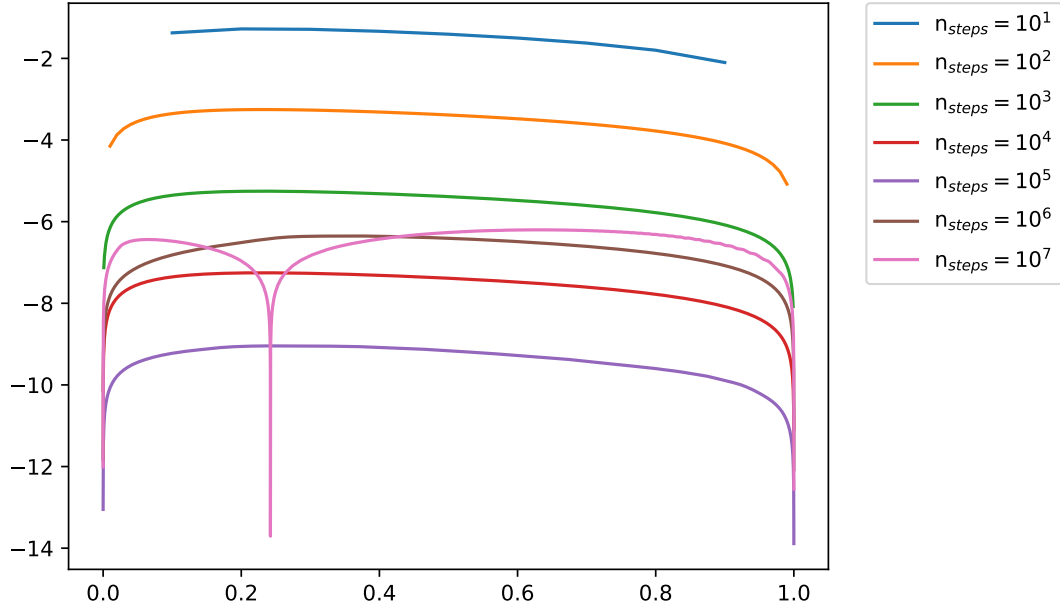


FIG. 3. Absolute error for different step sizes n_{steps} .

b)

Increasing number of steps also result in a decrease of absolute error.

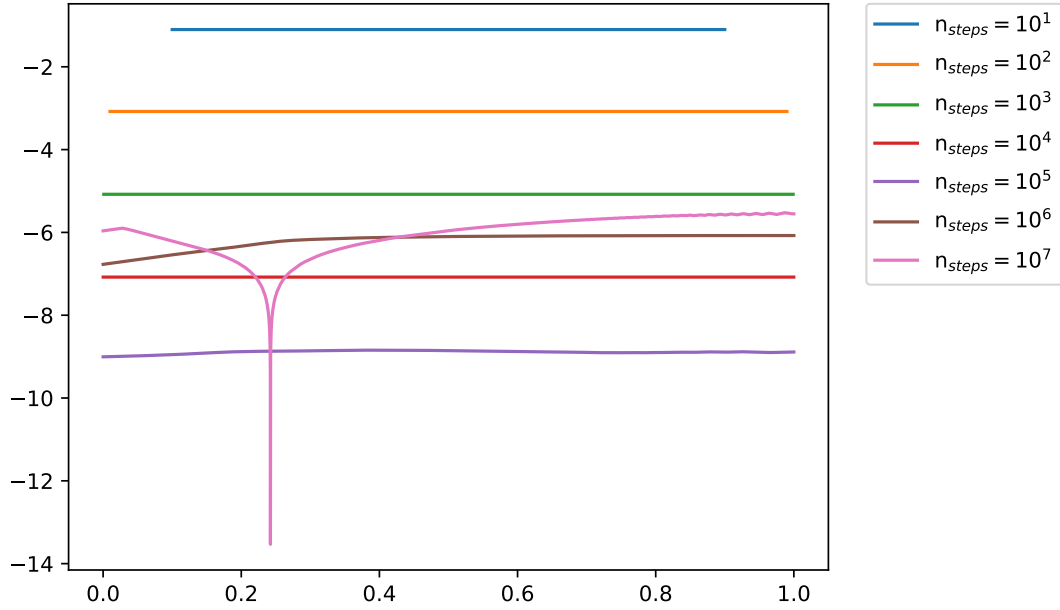


FIG. 4. Relative error for different step sizes n_{steps} .

c)

Increasing number of steps result in a decrease of maximum relative error, up to a certain number of steps. At $n_{steps} \approx 10^5$ the maximum relative error increase. This can be related to loss of numerical precision when step size is

small.

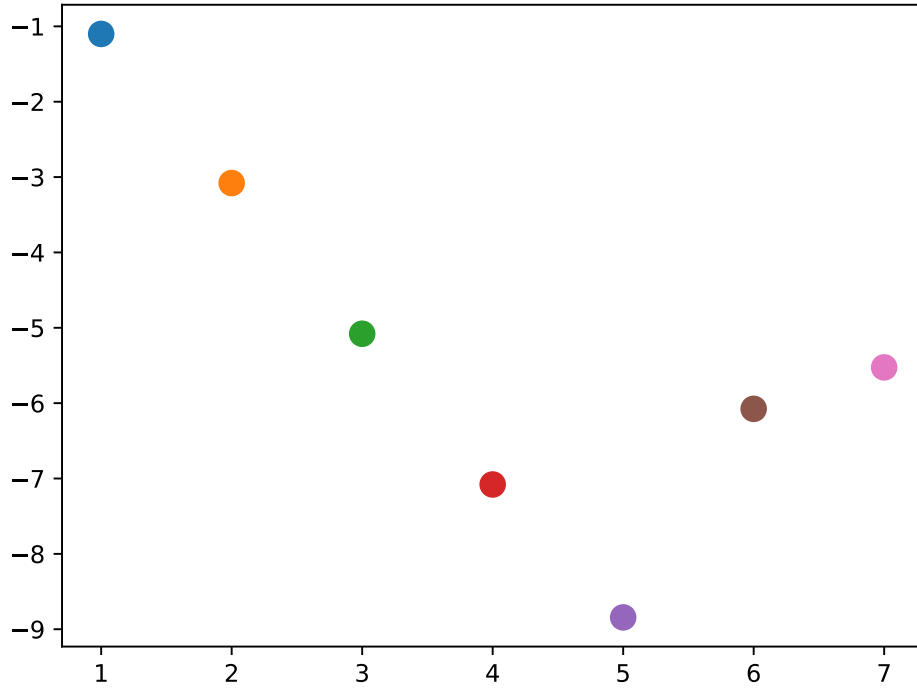


FIG. 5. Maximum relative error for each step sizes n_{steps} .

PROBLEM 9

a)

The special algorithm does not require the values of all a_i , b_i , c_i . We find the values of $\hat{b}_i$ from simplifying the general case

$$\hat{b}_i = b_i - \frac{a_i \cdot c_{i-1}}{\hat{b}_{i-1}}$$

$$\hat{b}_i = 2 - \frac{1}{\hat{b}_{i-1}}$$

Calculating the first values to see a pattern

$$\hat{b}_1 = 2$$

$$\hat{b}_2 = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\hat{b}_3 = 2 - \frac{1}{\frac{3}{2}} = \frac{4}{3}$$

$$\hat{b}_4 = 2 - \frac{1}{\frac{4}{3}} = \frac{5}{4}$$

$\vdots$

$$\hat{b}_i = \frac{i+1}{i}$$

for $i = 1, 2, \dots, n$

Algorithm 2 Special algorithm

procedure FORWARD SWEEP($\vec{b}$)

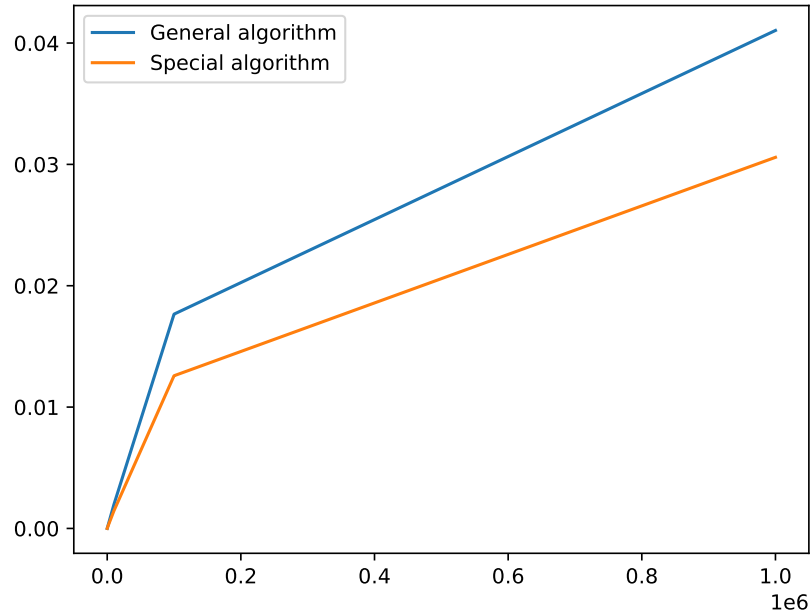
 $n \leftarrow \text{length of } \vec{b}$
 $\vec{\hat{b}}, \vec{\hat{g}} \leftarrow \text{vectors of length } n.$
 $\hat{b}_1 \leftarrow 2$
 $\triangleright$ Handle first element in main diagonal outside loop

 $\hat{g}_1 \leftarrow g_1$
for $i = 2, 3, \dots, n$ **do**
 $\hat{b}_i \leftarrow \frac{i+1}{i}$
 $\hat{g}_i \leftarrow g_i + \frac{\hat{g}_{i-1}}{\hat{b}_{i-1}}$
return $\vec{\hat{b}}, \vec{\hat{g}}$
procedure BACKWARD SWEEP($\vec{\hat{b}}, \vec{\hat{g}}$)

 $n \leftarrow \text{length of } \vec{\hat{b}}$
 $\vec{v} \leftarrow \text{vector of length } n.$
 $v_n \leftarrow \frac{\hat{g}_n}{\hat{b}_n}$
for $i = n - 1, n - 2, \dots, 1$ **do**
 $v_i \leftarrow \frac{\hat{g}_i + v_{i+1}}{\hat{b}_i}$
return $\vec{v}$

b)

For every iteration of i in forward sweep we have 2 divisions, and 2 additions, resulting in $4(n - 1)$ FLOPs. For backward sweep we have 1 division, and for every iteration of i we have 1 addition, and 1 division, resulting in $2(n - 1) + 1$ FLOPs. Total FLOPs for the special algorithm is $6(n - 1) + 1$.

PROBLEM 10

 FIG. 6. Timing of general algorithm vs. special for step sizes n_{steps}