Project 1

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https://github.uio.no/FYS3150-G2-2023/Project-1

PROBLEM 1

$$\begin{aligned}
u(x) &= \int\int \frac{d^2u}{dx^2} dx^2 \\
&= \int\int -100e^{-10x} dx^2 \\
&= \int \frac{-100e^{-10x}}{-10} + c_1 dx \\
&= \int 10e^{-10x} + c_1 dx \\
&= \frac{10e^{-10x}}{-10} + c_1 x + c_2 \\
&= -e^{-10x} + c_1 x + c_2
\end{aligned}$$

Using the boundary conditions, we can find c_1 and c_2 as shown below:

$$\begin{aligned}
u(0) &= 0 \\
-e^{-10\cdot 0} + c_1 \cdot 0 + c_2 &= 0 \\
-1 + c_2 &= 0 \\
c_2 &= 1
\end{aligned}$$

$$\begin{aligned}
u(1) &= 0 \\
-e^{-10\cdot 1} + c_1 \cdot 1 + c_2 &= 0 \\
-e^{-10} + c_1 + c_2 &= 0 \\
c_1 &= e^{-10} - c_2 \\
c_1 &= e^{-10} - 1
\end{aligned}$$

Using the values that we found for c_1 and c_2, we get

$$\begin{aligned}
u(x) &= -e^{-10x} + (e^{-10} - 1)x + 1 \\
&= 1 - (1 - e^{-10}) - e^{-10x}
\end{aligned}$$

PROBLEM 2

PROBLEM 3

To derive the discretized version of the Poisson equation, we first need the taylor expansion for $u(x)$ around $x + h$ and $x - h$.

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + \mathcal{O}(h^4)$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + \mathcal{O}(h^4)$$

If we add the equations above, we get this new equation:

$$\begin{aligned}
u(x+h) + u(x-h) &= 2u(x) + u''(x)h^2 + \mathcal{O}(h^4) \\
u(x+h) - 2u(x) + u(x-h) + \mathcal{O}(h^4) &= u''(x)h^2 \\
u''(x) &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2) \\
u_i''(x) &= \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \mathcal{O}(h^2)
\end{aligned}$$

We can then replace $\frac{d^2u}{dx^2}$ with the RHS (right-hand side) of the equation:

$$\begin{aligned}
-\frac{d^2u}{dx^2} &= 100e^{-10x} \\
\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + \mathcal{O}(h^2) &= 100e^{-10x}
\end{aligned}$$

And lastly, we leave out $\mathcal{O}(h^2)$ and change u_i to v_i to differentiate between the exact solution and the approximate solution, and get the discretized version of the equation:

$$align* \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} = 100e^{-10x}$$

PROBLEM 4

The value of $u(x_0)$ and $u(x_1)$ is known, using the discretized equation we can approximate the value of $f(x_i) = f_i$. This will result in a set of equations

$$\begin{gathered}
-v_0 + 2v_1 - v_2 = h^2 \cdot f_1 \\
-v_1 + 2v_2 - v_3 = h^2 \cdot f_2 \\
\vdots \\
-v_{m-2} + 2v_{m-1} - v_m = h^2 \cdot f_{m-1}
\end{gathered}$$

Rearranging the first and last equation, moving terms of known boundary values to the RHS

$$\begin{aligned}
2v_1 - v_2 &= h^2 \cdot f_1 + v_0 \\
-v_1 + 2v_2 - v_3 &= h^2 \cdot f_2 \\
&\vdots \\
-v_{m-2} + 2v_{m-1} &= h^2 \cdot f_{m-1} + v_m
\end{aligned}$$

We now have a number of linear eqations, corresponding to the number of unknown values, which can be represented as an augmented matrix

$$\left[\begin{array}{ccccc|c} 2v_1 & -v_2 & 0 & \dots & 0 & g_1 \\ -v_1 & 2v_2 & -v_3 & 0 & & g_2 \\ 0 & -v_2 & 2v_3 & -v_4 & & g_2 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & & & -v_{m-2} & 2v_{m-1} & g_{m-1} \end{array}\right]$$

where $g_i = h^2 f_i$. An augmented matrix can be represented as $\boldsymbol{A}\vec{x} = \vec{b}$. In this case $\boldsymbol{A}$ is the coefficient matrix with a tridiagonal signature $(-1, 2, -1)$ and dimension $n \times n$, where $n = m - 2$.

PROBLEM 5

a)

b)

PROBLEM 6

a)

b)

PROBLEM 7

PROBLEM 8

PROBLEM 9