Project 1

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https://github.uio.no/FYS3150-G2-2023/Project-1

PROBLEM 1

$$
\begin{aligned}
u(x) &= \int\int \frac{d^2u}{dx^2}dx^2 \\
&= \int\int -100e^{-10x}dx^2 \\
&= \int \frac{-100e^{-10x}}{-10} + c_1 dx \\
&= \int 10e^{-10x} + c_1 dx \\
&= \frac{10e^{-10x}}{-10} + c_1 x + c_2 \\
&= -e^{-10x} + c_1 x + c_2
\end{aligned}
$$

Using the boundary conditions, we can find c_1 and c_2 as shown below:

$$
\begin{aligned}
u(0) &= 0 \\
-e^{-10\cdot 0} + c_1 \cdot 0 + c_2 &= 0 \\
-1 + c_2 &= 0 \\
c_2 &= 1
\end{aligned}
$$

$$
\begin{aligned}
u(1) &= 0 \\
-e^{-10\cdot 1} + c_1 \cdot 1 + c_2 &= 0 \\
-e^{-10} + c_1 + c_2 &= 0 \\
c_1 &= e^{-10} - c_2 \\
c_1 &= e^{-10} - 1
\end{aligned}
$$

Using the values that we found for c_1 and c_2, we get

$$
\begin{aligned}
u(x) &= -e^{-10x} + (e^{-10} - 1)x + 1 \\
&= 1 - (1 - e^{-10}) - e^{-10x}
\end{aligned}
$$

PROBLEM 2

PROBLEM 3

To derive the discretized version of the Poisson equation, we first need the taylor expansion for $u(x)$ around $x + h$ and $x - h$.

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + \mathcal{O}(h^4)$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + \mathcal{O}(h^4)$$

If we add the equations above, we get this new equation:

$$\begin{aligned} u(x+h) + u(x-h) &= 2u(x) + u''(x)h^2 + \mathcal{O}(h^4) \\ u(x+h) - 2u(x) + u(x-h) + \mathcal{O}(h^4) &= u''(x)h^2 \\ u''(x) &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2) \\ u_i''(x) &= \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \mathcal{O}(h^2) \end{aligned}$$

We can then replace $\frac{d^2u}{dx^2}$ with the RHS (right-hand side) of the equation:

$$\begin{aligned} -\frac{d^2u}{dx^2} &= 100e^{-10x} \\ \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + \mathcal{O}(h^2) &= 100e^{-10x} \end{aligned}$$

And lastly, we leave out $\mathcal{O}(h^2)$ and change u_i to v_i to differentiate between the exact solution and the approximate solution, and get the discretized version of the equation:

$$align * \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} = 100e^{-10x}$$

PROBLEM 4

The value of $u(x_0)$ and $u(x_1)$ is known, using the discretized equation we solve for $\vec{v}$. This will result in a set of equations

$$\begin{aligned} -v_0 + 2v_1 - v_2 &= h^2 \cdot f_1 \\ -v_1 + 2v_2 - v_3 &= h^2 \cdot f_2 \\ &\vdots \\ -v_{m-2} + 2v_{m-1} - v_m &= h^2 \cdot f_{m-1} \end{aligned}$$

where $v_i = v(x_i)$ and $f_i = f(x_i)$. Rearranging the first and last equation, moving terms of known boundary values to the RHS

$$\begin{aligned} 2v_1 - v_2 &= h^2 \cdot f_1 + v_0 \\ -v_1 + 2v_2 - v_3 &= h^2 \cdot f_2 \\ &\vdots \\ -v_{m-2} + 2v_{m-1} &= h^2 \cdot f_{m-1} + v_m \end{aligned}$$

We now have a number of linear eqations, corresponding to the number of unknown values, which can be represented as an augmented matrix

$$\left[\begin{array}{ccccc|c} 2v_1 & -v_2 & 0 & \dots & 0 & g_1 \\ -v_1 & 2v_2 & -v_3 & 0 & & g_2 \\ 0 & -v_2 & 2v_3 & -v_4 & & g_2 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & & & -v_{m-2} & 2v_{m-1} & g_{m-1} \end{array}\right]$$

Since the boundary values are equal to 0 the RHS can be renamed $g_i = h^2 f_i$ for all i. An augmented matrix can be represented as $\boldsymbol{A}\vec{x} = \vec{b}$. In this case $\boldsymbol{A}$ is the coefficient matrix with a tridiagonal signature $(-1, 2, -1)$ and dimension $n \times n$, where $n = m - 2$.

PROBLEM 5

a)

b)

PROBLEM 6

a)

Renaming the sub-, main-, and supdiagonal of matrix $\boldsymbol{A}$

$$\vec{a} = [a_2, a_3, ..., a_{n-1}, a_n]$$
$$\vec{b} = [b_1, b_2, b_3, ..., b_{n-1}, b_n]$$
$$\vec{c} = [c_1, c_2, c_3, ..., c_{n-1}]$$

Following Thomas algorithm for gaussian elimination, we first perform a forward sweep

Algorithm 1 Foreward sweep

Create new vector of length n.
[0] = b[0] ▷ Handle first element in main diagonal outside loop
for $i = 1, ..., n-1$ **do**
    d = a[i-1] $\overline{b[i-1] \quad b-=d*(*sup_d iag)(i-1);(*g_v ec)(i)-=d*(*g_v ec)(i-1);}$
        while Some condition **do**
            Do something more here
        Maybe even some more math here, e.g $\int_0^1 f(x)\mathrm{d}x$

Here the index i does not refer to the element in the vector, ie. b_i, but to the index in the vector where the element is found ie. $b[i] = b_{i+1}$.

$$\hat{b}_1 = b_1$$
$$\hat{b}_2 = b_2 - \frac{a_2}{\hat{b}_1} \cdot c_1$$
$$\vdots$$
$$\hat{b}_i = b_i - \frac{a_i}{\hat{b}_{i-1}} \cdot c_{i-1}$$
$$\vdots$$

b)

PROBLEM 7

PROBLEM 8

PROBLEM 9