Project 3

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Add an abstract for the project?

I. INTRODUCTION

We are surrounded by matter, which are made up of elementary particles. In the field of physics we want to understand the properties of these particles, measure their physical quantities, not to mention explain the origin of mass [?].

However, to study a particle, it is necessary isolate and contain it over time. The Penning trap is a device, able to confine charged particles for a period of time. This concept was evolved from F. M. Penning's implementation of magnetic fiels to a vaccum gauge, and J. R. Pierce's work with electron beams, and put into practice by Hans Dehmelt. In 1973 Dehmelt and his group of researchers were able to confine a particle and store it over several months [?].

In practice, a Penning trap is not easy to obtain, and an experiment is both time consuming and expensive. A numerical approach, allow us to study the effects of the Penning trap on a charged particle, without the cost. We can use ordinary differential equations to model the particle's movement, confined within an Penning trap. Our focus will be on an ideal Penning trap, where an electrostatic field confines the particle in z-direction, and a magnetic field confines it in the radial direction. We will use numerical methods to model a single particle, and study the particle motion in radial direction. In addition, we will model a system of particles, and study their motion both with and without particle interaction.

II. THEORY AND METHODS

When we study the Penning traps effect on a particle with a charge q, we need to consider the forces acting on the particle. We can use Newton's second law (**??**) to determine the position of a particle as a function of time. In addition, we introduce the Lorentz force (**??**), which describes the force on the particle. The position can be described by

$$m\ddot{\mathbf{r}} = (q\mathbf{E} + q\mathbf{v} \times \mathbf{B}). \tag{1}$$

Using eq. (**??**) we derive the differential equations in sec. **??**, and can rewrite them as

$$\ddot{x} - \omega_0 \dot{y} - \frac{1}{2}\omega_z^2 x = 0, \tag{2}$$

$$\ddot{y} - \omega_0 \dot{x} - \frac{1}{2}\omega_z^2 y = 0, \tag{3}$$

$$\ddot{z} + \omega_z^2 z = 0, \tag{4}$$

We find the general solution for eq. (**??**)

$$z(t) = c_1 e^{i\omega_z t} + c_2 e^{-i\omega_z t}, \tag{5}$$

derived in sec. **??**. Continuing, we will use a Calcium ion with a single positive charge. That is, we assume the charge of the particle is $q > 0$.

Since eq. (**??**) and eq. (**??**) are coupled, we want to rewrite them as a single differential equation. We derive this in sec. **??**, and the resulting equation is given by

$$\ddot{f} + i\omega_0 \dot{f} - \frac{1}{2}\omega_z^2 f = 0. \tag{6}$$

Eq. (**??**) has a general solution given by

$$f(t) = A_+ e^{-i(\omega_+ t + \phi_+)} + A_- e^{-i(\omega_- t + \phi_-)}. \tag{7}$$

The amplitude A_+ and A_- are positive, the phases ϕ_+ and ϕ_- are constant, and the rate is given by

$$\omega_\pm = \frac{\omega_0 \pm \sqrt{\omega_0^2 - 2\omega_z^2}}{2}.$$

We find the physical coordinates at a given time t using

$$x(t) = \mathrm{Re} f(t), \quad y(t) = \mathrm{Im} f(t),$$

and eq. (**??**). We can rearrange the right hand side of the derived expression in **??**, and find the physical coordinates

$$x(t) = A_+ \cos(\omega_+ t + \phi_+) + A_- \cos(\omega_- t + \phi_-) \tag{8}$$

$$y(t) = -A_+ \sin(\omega_+ t + \phi_+) - A_- \sin(\omega_- t + \phi_-) \tag{9}$$

However, to obtain a bound on the particle's movement in the radial direction, we have to ensure that

$$|f(t)| = \sqrt{(x(t))^2 + (y(t))^2}.$$

When $t \to \infty$, the upper and lower limits are

$$R_+ = A_+ + A_- \tag{10}$$

$$R_- = |A_+ - A_-|, \tag{11}$$

derived in sec. **??**.

To obtain the bounded solution, we need to consider the expression

$$\alpha = (\omega_+ - \omega_-)t + (\phi_+ - \phi_-).$$

When $t \to \infty$ the constant phases will not affect the expression, we obtain a bounded solution if we consider the rate such that $|f(t)| < \infty$. That is, we need $\omega_0^2 - 2\omega_z^2$ in eq. (??) to be real.

$$\omega_0^2 - 2\omega_z^2 = 0, \quad \omega_0 > 2\omega_z$$

$$\omega_\pm = \frac{\omega_0 \pm \sqrt{\omega_0^2 - 2\omega_z^2}}{2} \tag{12}$$

We find the solution Physical properties given by newtons second law (??)

The particle moves and its position can be determined using newton. where the electric field

Constant	Value	
B_0	$1.00T$	$9.65 \times 10^1 \frac{u}{(\mu s)e}$
V_0	$25.0mV$	$2.41 \times 10^6 \frac{u(\mu m)^2}{(\mu s)^2 e}$
d	$500\ \mu m$	

TABLE I. Default configuration of the Penning trap, where the value of T and V can be found in table ??.

A. Dealing with a multi–particle system

For a multi-particles system, we need to modify $\mathbf{F}$ to account for the force of other particles in the system acting upon each other. To do that, we add another term to $\mathbf{F}$

$$\mathbf{F}_i(t, \mathbf{v}_i, \mathbf{r}_i) = q_i \mathbf{E}(t, \mathbf{r}_i) + q_i \mathbf{v}_i \times \mathbf{B} - \mathbf{E}_p(t, \mathbf{r}_i), \tag{13}$$

where i and j are particle indices and

$$\mathbf{E}_p(t, \mathbf{r}_i) = q_i k_e \sum_{j \neq i} q_j \frac{\mathbf{r_i} - \mathbf{r_j}}{|\mathbf{r_i} - \mathbf{r_j}|^3}. \tag{14}$$

Newton's second law for a particle i is then

$$\frac{d^2 \mathbf{r}_i}{dt^2} = \frac{\mathbf{F}_i\left(t, \frac{d\mathbf{r}_i}{dt}, \mathbf{r_i}\right)}{m_i}, \tag{15}$$

We can then rewrite the second order ODE from equation ?? into a set of coupled first order ODEs. We now rewrite Newton's second law of motion as

$$\begin{aligned} \frac{d\mathbf{r}_i}{dt} &= \mathbf{v}_i \\ \frac{d\mathbf{v}_i}{dt} &= \frac{\mathbf{F}_i(t, \mathbf{v}_i, \mathbf{r}_i)}{m_i}. \end{aligned} \tag{16}$$

B. Forward Euler

For a particle i, the forward Euler method for a coupled system is expressed as

$$\begin{aligned} \mathbf{r}_{i,j+1} &= \mathbf{r}_{i,j} + h \cdot \frac{d\mathbf{r}_{i,j}}{dt} = \mathbf{r}_{i,j} + h \cdot \mathbf{v}_{i,j} \\ \mathbf{v}_{i,j+1} &= \mathbf{v}_{i,j} + h \cdot \frac{\mathbf{v}_{i,j}}{dt} = \mathbf{v}_{i,j} + h \cdot \frac{\mathbf{F}\left(t_j, \mathbf{v}_{i,j}, \mathbf{r}_{i,j}\right)}{m}, \end{aligned} \tag{17}$$

for particle i where j is the current time step of the particle, m is the mass of the particle, and h is the step length.

When dealing with a multi-particle system, we need to ensure that we do not update the position of any particles until every particle has calculated their next step. An easy way of doing this is to create a copy of all the particles, then update the copy, and when all the particles have calculated their next step, simply replace the particles with the copies. Algorithm ?? provides an overview on how that can be achieved.

Algorithm 1 Forward Euler method

```
procedure EVOLVE FORWARD EULER(particles, dt)
    N ← Number of particles in particles
    a ← Calculate F_i/m_i for each particle in particles
    for i = 1, 2, ..., N do
        particles_i.r ← particles_i.r + dt · particles_i.v
        particles_i.v ← particles_i.v + dt · a_i
```

C. 4th order Runge-Kutta

For a particle i, we can express the 4th order Runge-Kutta (RK4) method as

$$\begin{aligned} \mathbf{v}_{i,j+1} &= \mathbf{v}_{i,j} + \frac{h}{6}\left(\mathbf{k}_{\mathbf{v},1,i} + 2\mathbf{k}_{\mathbf{v},2,i} + 2\mathbf{k}_{\mathbf{v},3,i} + \mathbf{k}_{\mathbf{v},4,i}\right) \\ \mathbf{r}_{i,j+1} &= \mathbf{r}_{i,j} + \frac{h}{6}\left(\mathbf{k}_{\mathbf{r},1,i} + 2\mathbf{k}_{\mathbf{r},2,i} + 2\mathbf{k}_{\mathbf{r},3,i} + \mathbf{k}_{\mathbf{r},4,i}\right), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mathbf{k}_{\mathbf{v},1,i} &= \frac{\mathbf{F}_i(t_j, \mathbf{v}_{i,j}, \mathbf{r}_{i,j})}{m} \\ \mathbf{k}_{\mathbf{r},1,i} &= \mathbf{v}_{i,j} \\ \mathbf{k}_{\mathbf{v},2,i} &= \frac{\mathbf{F}_i(t_j + \frac{h}{2}, \mathbf{v}_{i,j} + h \cdot \frac{\mathbf{k}_{\mathbf{v},1,i}}{2}, \mathbf{r}_{i,j} + h \cdot \frac{\mathbf{k}_{\mathbf{r},1,i}}{2})}{m} \\ \mathbf{k}_{\mathbf{r},2,i} &= \mathbf{v}_{i,j} + h \cdot \frac{\mathbf{k}_{\mathbf{v},1,i}}{2} \\ \mathbf{k}_{\mathbf{v},3,i} &= \frac{\mathbf{F}_i(t_j + \frac{h}{2}, \mathbf{v}_{i,j} + h \cdot \frac{\mathbf{k}_{\mathbf{v},2,i}}{2}, \mathbf{r}_{i,j} + h \frac{\cdot \mathbf{k}_{\mathbf{r},2,i}}{2})}{m} \\ \mathbf{k}_{\mathbf{r},3,i} &= \mathbf{v}_{i,j} + h \cdot \frac{\mathbf{k}_{\mathbf{v},2,i}}{2} \\ \mathbf{k}_{\mathbf{v},4,i} &= \frac{\mathbf{F}_i(t_j + h, \mathbf{v}_{i,j} + h \cdot \mathbf{k}_{\mathbf{v},3,i}, \mathbf{r}_{i,j} + h \cdot \mathbf{k}_{\mathbf{r},3,i})}{m} \\ \mathbf{k}_{\mathbf{r},4,i} &= \mathbf{v}_{i,j} + h \cdot \frac{\mathbf{k}_{\mathbf{v},1,i}}{2}. \end{aligned} \tag{19}$$

In order to find each $\mathbf{k}_{\mathbf{r},i}$ and $\mathbf{k}_{\mathbf{v},i}$, we need to first compute all $\mathbf{k}_{\mathbf{r},i}$ and $\mathbf{k}_{\mathbf{v},i}$ for all particles, then we can update the particles in order to compute $\mathbf{k}_{\mathbf{r},i+1}$ and $\mathbf{k}_{\mathbf{v},i+1}$. In order for the algorithm to work, we need to save a copy of each particle before starting so that we can update the particles correctly for each step.

Algorithm 2 RK4 method

procedure EVOLVE RK4($particles, dt$)
    $N \leftarrow$ Number of particles inside the Penning trap
    $orig\_p \leftarrow$ Copy of particles
    $tmp\_p \leftarrow$ Array of particles of size N
    $\mathbf{k_r} \leftarrow$ 2D array of vectors of size $4 \times N$
    $\mathbf{k_v} \leftarrow$ 2D array of vectors of size $4 \times N$
    for $i = 1, 2, \ldots, N$ **do**
        $\mathbf{k}_{\mathbf{r},1,i} \leftarrow particles_i.\mathbf{v}$
        $\mathbf{k}_{\mathbf{v},1,i} \leftarrow \frac{\mathbf{F}_i}{m_i}$
        $tmp\_p_i.\mathbf{r} \leftarrow orig\_p_i.\mathbf{r} + \frac{dt}{2} \cdot \mathbf{k}_{\mathbf{r},1,i}$
        $tmp\_p_i.\mathbf{v} \leftarrow orig\_p_i.\mathbf{v} + \frac{dt}{2} \cdot \mathbf{k}_{\mathbf{v},1,i}$
    $particles \leftarrow tmp\_p$ ▷ Update particles
    for $i = 1, 2, \ldots, N$ **do**
        $\mathbf{k}_{\mathbf{r},2,i} \leftarrow particles_i.\mathbf{v}$
        $\mathbf{k}_{\mathbf{v},2,i} \leftarrow \frac{\mathbf{F}_i}{m_i}$
        $tmp\_p_i.\mathbf{r} \leftarrow orig\_p_i.\mathbf{r} + \frac{dt}{2} \cdot \mathbf{k}_{\mathbf{r},2,i}$
        $tmp\_p_i.\mathbf{v} \leftarrow orig\_p_i.\mathbf{v} + \frac{dt}{2} \cdot \mathbf{k}_{\mathbf{v},2,i}$
    $particles \leftarrow tmp\_p$ ▷ Update particles
    for $i = 1, 2, \ldots, N$ **do**
        $\mathbf{k}_{\mathbf{r},3,i} \leftarrow particles_i.\mathbf{v}$
        $\mathbf{k}_{\mathbf{v},3,i} \leftarrow \frac{\mathbf{F}_i}{m}$
        $tmp\_p_i.\mathbf{r} \leftarrow orig\_p_i.\mathbf{r} + dt \cdot \mathbf{k}_{\mathbf{r},3,i}$
        $tmp\_p_i.\mathbf{v} \leftarrow orig\_p_i.\mathbf{v} + dt \cdot \mathbf{k}_{\mathbf{v},3,i}$
    $particles \leftarrow tmp\_p$ ▷ Update particles
    for $i = 1, 2, \ldots, N$ **do**
        $\mathbf{k}_{\mathbf{r},4,i} \leftarrow particles_i.\mathbf{v}$
        $\mathbf{k}_{\mathbf{v},4,i} \leftarrow \frac{\mathbf{F}}{m}$
        $tmp\_p_i.\mathbf{r} \leftarrow orig\_p_i.\mathbf{r} + \frac{dt}{6} \cdot (\mathbf{k}_{\mathbf{r},1,i} + \mathbf{k}_{\mathbf{r},2,i} + \mathbf{k}_{\mathbf{r},3,i} + \mathbf{k}_{\mathbf{r},4,i})$
        $tmp\_p_i.\mathbf{v} \leftarrow orig\_p_i.\mathbf{v} + \frac{dt}{6} \cdot (\mathbf{k}_{\mathbf{v},1,i} + \mathbf{k}_{\mathbf{v},2,i} + \mathbf{k}_{\mathbf{v},3,i} + \mathbf{k}_{\mathbf{v},4,i})$
    $particles \leftarrow tmp\_p$ ▷ Final update

$\mathbf{F}$ in the algorithm does not take any arguments as it uses the velocities and positions of the particles inside the array *particles* to calculate the total force acting on particle i.

This approach would require 8 loops to be able to complete the calculation since we cannot update the particles until after all $\mathbf{k}$ values have been computed, however if we create a temporary array that holds particles, we can put the updated particles in there, and then use that array in the next loop, and would reduce the required amount of loops down to 4.

D. Testing the simulation

E. Relative error and error convergance rate

F. Tools

The numerical methods implemented in C++, are parallelized using OpenMP [**?**]. We used the Python library matplotlib [**?**] to produce all the plots, and seaborn [**?**] to set the theme in the figures.

III. RESULTS AND DISCUSSION

We simulated the movement of particles confined in a Penning trap. All simulations used the initial conditions for particle 1 and 2 given in table **??**.

First we simulated a single particle for $50\mu s$, approximating the particle's motion using the RK4 method. In addition we compared the motion of particle 1 with the analytical solution in figure **??**. What we see is a complete overlap of the analytical solution completely overlap the approximated, suggest that the simulation result is good.

Particle	Position	Velocity
p_1	$(20, 0, 20)\mu m$	$(0, 25, 0)\mu m/\mu s$
p_2	$(25, 25, 0)\mu m$	$(0, 40, 5)\mu m/\mu s$

TABLE II. Initial position and velocity of particle 1 (p_1) particle 2 (p_2), where the analytical solution is given by $z(t) = z_0 \cos(\omega_z t)$

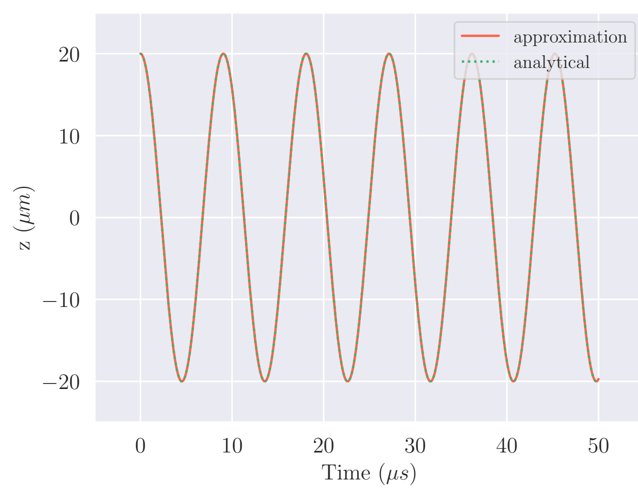

FIG. 1. A single particle in the Penning trap, approximated and analytical motion in z-direction.

We will now consider the Penning trap with initial conditions given in table **??**, and simulate using one or two particles. In addition, we simulate two particles both with and without interactions, the result is found in figure **??**. When we add interaction between the particles, they both still follow the same inherent path. However, we observe a small shift in both particle's movement.

B_0	V_0	d

TABLE III. Caption

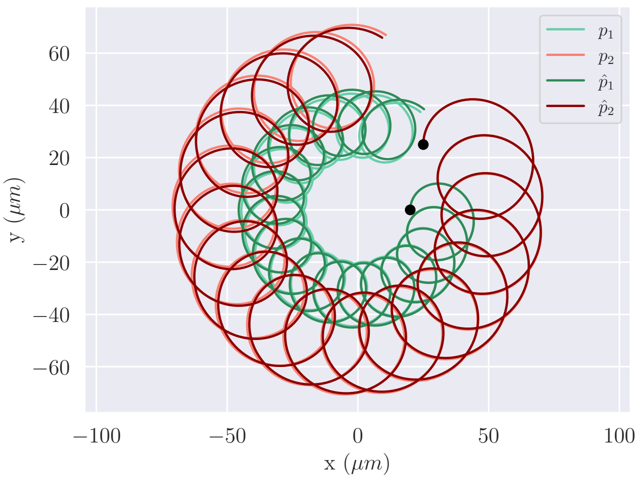

FIG. 2. Movement of two particles in the xy-plane. $\hat{p}_1$ and $\hat{p}_2$ include particle interaction, whereas p_1 and p_2 does not include particle interaction.

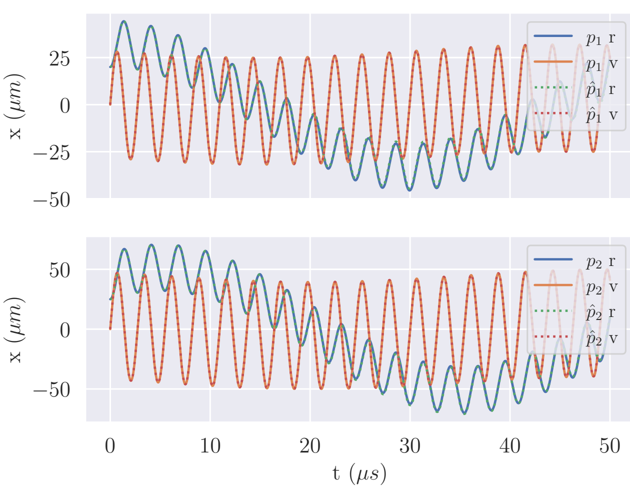

FIG. 3. Phase space plot of two particles in x-direction.

When we simulate two particles, we can see the effect of interaction between the particles in the xy-plane in fig. **??** and in the z-direction in fig. **??**. What we observe is a very small shift in position for particle 1 in x-direction, whereas particle 2 does not have a visible shift. In the z-direction, however, the oscillation of particle 2 experience a greater shift. Particle 2 experience the force of particle 1 such that particle 2 moves larger distance.

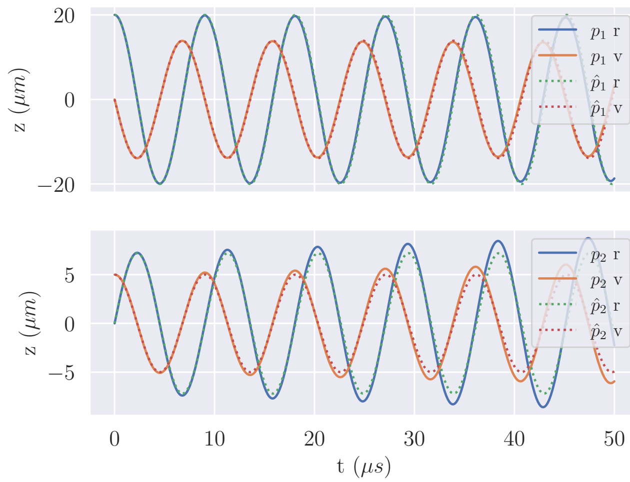

FIG. 4. Phase space plot of two particles in z-direction.

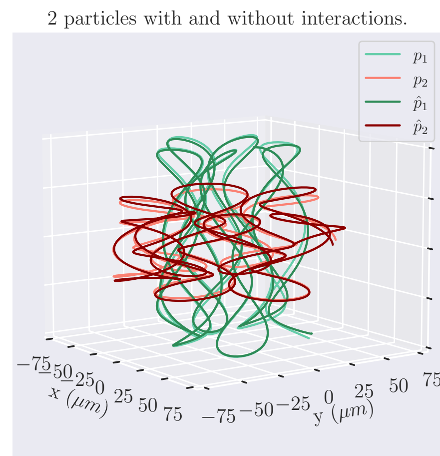

FIG. 5. 3D plot of particles-

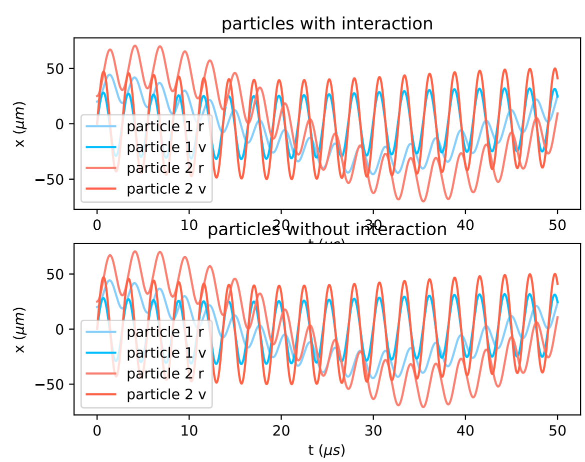

FIG. 6. Phase space plot of two particles.

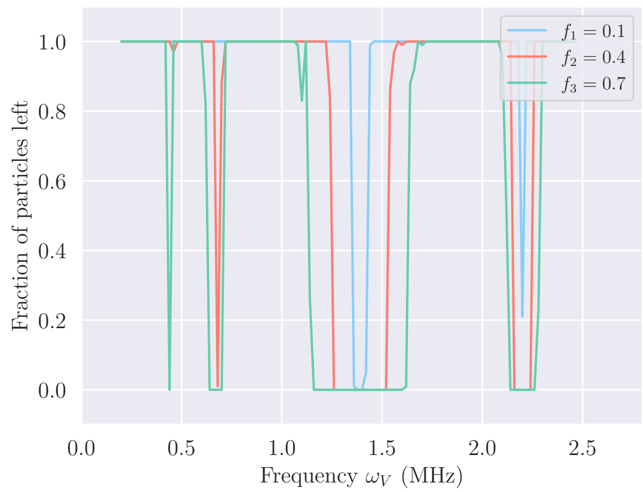

FIG. 7. Fraction of particles left in the Penning trap, with a given amplitude f.

IV. CONCLUSION

Appendix A: Equations of electrodynamics and classical mechanics

Newton's second law

$$m\ddot{\mathbf{r}} = \sum_i \mathbf{F}_i \tag{A1}$$

Lorentz force

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}, \tag{A2}$$

Electric field

$$\mathbf{E} = -\nabla V \tag{A3}$$

Electric field at a point $\mathbf{r}$

$$\mathbf{E} = k_e \sum_{j=1}^{n} q_j \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|^3} \tag{A4}$$

Appendix B: Derivation of equations

1. Equations of motion

First, we need to define the velocity of the particle

$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

We can rewrite the velocity as $\dot{r} = (\dot{x}, \dot{y}, \dot{z})$, and find the cross product

$$q\mathbf{v} \times \mathbf{B} = q \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B_0 \end{vmatrix} = q\big(B_0\dot{y}, -B_0\dot{x}, 0\big).$$

We are considering an ideal Penning traps, where we define the electric potential as

$$V(x, y, z) = \frac{V_0}{2d^2}(2z^2 - x^2 - y^2).$$

The relationship between the electric field $\mathbf{E}$ and the electric potential of the field is given by

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= -\left(\frac{dV}{dx}, \frac{dV}{dy}\frac{dV}{dz}\right) \\ &= \frac{V_0}{d^2}\big(x, y, -2z\big). \end{aligned}$$

We can now express the Lorentz force as

$$\begin{aligned} \mathbf{F} &= q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \\ &= \frac{qV_0}{d^2}\big(x, y, -2z\big) + \big(qB_0\dot{y}, -qB_0\dot{x}, 0\big), \end{aligned}$$

and insert it into Newtons equation (**??**). We get

$$\ddot{\mathbf{r}} = \left(\frac{qV_0}{md^2}x, \frac{qV_0}{md^2}y, -\frac{2qV_0}{md^2}z\right) + \left(\frac{qB_0}{m}\dot{y}, -\frac{qB_0}{m}\dot{x}, 0\right),$$

which can be written as

$$\begin{aligned} \ddot{x} &= \frac{qV_0}{md^2}x + \frac{qB_0}{m}\dot{y}, \\ \ddot{y} &= \frac{qV_0}{md^2}y - \frac{qB_0}{m}\dot{x}, \\ \ddot{z} &= -\frac{2qV_0}{md^2}z. \end{aligned}$$

If we define

$$\omega_0 \equiv \frac{qB_0}{m}, \quad \omega_z^2 \equiv \frac{2qV_0}{md^2},$$

the equations of motion can be written as

$$\begin{aligned} \ddot{x} &= \frac{1}{2}\omega_z^2 x + \omega_0\dot{y}, \\ \ddot{y} &= \frac{1}{2}\omega_z^2 y - \omega_0\dot{x}, \\ \ddot{z} &= -\omega_z^2 z. \end{aligned}$$

2. General solution

We consider the characteristic equation of a second order differential equation [**?**],

$$\begin{aligned} r^2 + \omega_z^2 &= 0 \\ r &= \pm\sqrt{-\omega_z^2}. \end{aligned}$$

The characteristic equation has two complex roots

$$r_1 = -i\omega_z, \quad r_2 = i\omega_z,$$

which give us solutions in the general form

$$z = c_1 e^{i\omega_z t} + c_2 e^{-i\omega_z t}.$$

In addition, for a complex number $z = a + ib$, we can define $e^z \equiv e^a(\cos b + i \sin b)$ [?]. We can rewrite the general solution as

$$\begin{aligned} c_1 e^{i\omega_z t} + c_2 e^{-i\omega_z t} &= c_1(\cos\omega_z t + i\sin\omega_z t) \\ &\quad + c_2(\cos\omega_z t - i\sin\omega_z t \\ &= E\cos\omega_z t + iF\sin\omega_z t. \end{aligned}$$

3. Complex function

In sec. **??** we found the differential equations for $\ddot{x}$ and $\ddot{y}$. To derive a single differential equation, we introduce the complex function $f(t) = x(t) + iy(t)$, which gives us

$$\begin{aligned} 0 &= \Big(\ddot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x\Big) + i\Big(\ddot{y} + \omega_0\dot{x} - \frac{1}{2}\omega_z^2 y\Big) \\ &= \ddot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x + i\ddot{y} + i\omega_0\dot{x} - i\frac{1}{2}\omega_z^2 y \\ &= \ddot{x} + i\ddot{y} + i\omega_0\dot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x - i\frac{1}{2}\omega_z^2 y. \end{aligned}$$

Using the definition $i = \sqrt{-1}$, we can rewrite

$$i\omega_0\dot{x} + (-1)\omega_0\dot{y} = i\omega_0\dot{x} + i^2\omega_0\dot{y}.$$

This gives us a single differential equation

$$\begin{aligned} 0 &= \ddot{x} + i\ddot{y} + i\omega_0(\dot{x} + i\dot{y}) - \frac{1}{2}\omega_z^2 x - i\frac{1}{2}\omega_z^2 y \\ &= \ddot{f} + i\omega_0\dot{f} - \frac{1}{2}\omega_z^2 f. \end{aligned}$$

4. Physical coordinates

We can rewrite eq. (**??**), as

$$\begin{aligned} f(t) &= A_+ e^{-i(\omega_+ t + \phi_+)} + A_- e^{-i(\omega_- t + \phi_-)} \\ &= A_+(\cos(\omega_+ t + \phi_+) - i\sin(\omega_+ t + \phi_+)) \\ &\quad + A_-(\cos(\omega_- t + \phi_-) - i\sin(\omega_- t + \phi_-)). \end{aligned}$$

5. Upper and lower bounds

To obtain the upper and lower bounds of the particle's distance from the origin, we first find an expression for the second norm (?) defined as $|f(t)| = \sqrt{(x(t))^2 + (y(t))^2}$.

$$\begin{aligned} (x(t))^2 &= \big(A_+\cos(\omega_+ t + \phi_+) + A_-\cos(\omega_- t + \phi_-)\big)^2 \\ &= A_+^2\cos^2(\omega_+ t + \phi_+) \\ &\quad + 2A_+A_-\cos(\omega_+ t + \phi_+)\cos(\omega_- t + \phi_-) \\ &\quad + A_-^2\cos^2(\omega_- t + \phi_-), \end{aligned}$$

$$\begin{aligned} (y(t))^2 &= \big(-A_+\sin(\omega_+ t + \phi_+) - A_-\sin(\omega_- t + \phi_-)\big)^2 \\ &= A_+^2\sin^2(\omega_+ t + \phi_+) \\ &\quad + 2A_+A_-\sin(\omega_+ t + \phi_+)\sin(\omega_- t + \phi_-) \\ &\quad + A_-^2\sin^2(\omega_- t + \phi_-). \end{aligned}$$

We insert these expressions, and find

$$\begin{aligned} |f(t)| &= \sqrt{(x(t))^2 + (y(t))^2} \\ &= \sqrt{A_+^2 + 2A_+A_-\cos^2(\alpha) + A_-^2}, \end{aligned}$$

where $\alpha = (\omega_+ - \omega_-)t + (\phi_+ - \phi_-)$. If we set $\alpha = 0$ we get $\cos(0) = 1$, and obtain the upper bound

$$\begin{aligned} R_+ &= \sqrt{A_+^2 + 2A_+A_- + A_-^2} \\ &= \sqrt{(A_+ + A_-)^2} \\ &= A_+ + A_-. \end{aligned}$$

If $\alpha = \pi$ we get $\cos(\pi) = -1$, and find the lower bound

$$\begin{aligned} R_- &= \sqrt{A_+^2 - 2A_+A_- + A_-^2} \\ &= \sqrt{(A_+ - A_-)^2} \\ &= |A_+ - A_-|. \end{aligned}$$

6. Bounded solution

Appendix C: Numbers and units

Constant	Value	Unit
k_e (Coulomb)	1.38935333×10^5	$\frac{u(\mu m)^3}{(\mu s)^2 e^2}$
T (Tesla)	9.64852558×10^1	$\frac{u}{(\mu s)e}$
V (Volt)	9.64852558×10^7	$\frac{u(\mu m)^2}{(\mu s)^2 e}$

TABLE IV. Value of the Coulomb constant (k_e), and the SI units for magnetic field strength (T) and electric potential (V). The base units are given by length in micrometre (μm), time in microseconds (μs), mass in (u), and charge in elementary charge (e).

[1] The Editors of Encyclopaedia Britannica. matter. 2023.
[2] Manuel Vogel. *Particle Confinement in Penning Traps*. Springer Cham, 1–6 edition, 2018.
[3] Tom Lindstrøm. *Kalkulus*, pages 589–606. Universitetsforlaget, 4 edition, 2016.
[4] Tom Lindstrøm. *Kalkulus*, pages 133–137. Universitetsforlaget, 4 edition, 2016.